

ANTI-POWER PREFIXES OF THE THUE-MORSE WORD

COLIN DEFANT

*University of Florida
1400 Stadium Rd.
Gainesville, FL 32611 United States*

ABSTRACT. Recently, Fici, Restivo, Silva, and Zamboni defined a k -anti-power to be a word of the form $w_1w_2\cdots w_k$, where w_1, w_2, \dots, w_k are distinct words of the same length. They defined $AP(x, k)$ to be the set of all positive integers m such that the prefix of length km of the word x is a k -anti-power. Let \mathbf{t} denote the Thue-Morse word, and let $\mathcal{F}(k) = AP(\mathbf{t}, k) \cap (2\mathbb{Z}^+ - 1)$. For $k \geq 3$, $\gamma(k) = \min(\mathcal{F}(k))$ and $\Gamma(k) = \max((2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k))$ are well-defined odd positive integers. Fici et al. speculated that $\gamma(k)$ grows linearly in k . We prove that this is indeed the case by showing that $1/2 \leq \liminf_{k \rightarrow \infty} (\gamma(k)/k) \leq 9/10$ and $1 \leq \limsup_{k \rightarrow \infty} (\gamma(k)/k) \leq 3/2$. In addition, we prove that $\liminf_{k \rightarrow \infty} (\Gamma(k)/k) = 3/2$ and $\limsup_{k \rightarrow \infty} (\Gamma(k)/k) = 3$.

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1. INTRODUCTION

A well-studied notion in combinatorics on words is that of a k -power; this is simply a word of the form w^k for some word w . It is often interesting to ask questions related to whether or not certain types of words contain factors (also known as substrings) that are k -powers for some fixed k . For example, in 1912, Axel Thue [7] introduced an infinite binary word that does not contain any 3-powers as factors (we say such a word is cube-free). This infinite word is now known as the Thue-Morse word; it is arguably the world's most famous (mathematical) word [1, 2, 3, 4, 5].

Definition 1.1. Let \bar{w} denote the Boolean complement of a binary word w . Let $A_0 = 0$. For each nonnegative integer n , let $B_n = \bar{A}_n$ and $A_{n+1} = A_n B_n$. The *Thue-Morse word* \mathbf{t} is defined by

$$\mathbf{t} = \lim_{n \rightarrow \infty} A_n.$$

Recently, Fici, Restivo, Silva, and Zamboni [6] introduced the very natural concept of a k -anti-power; this is a word of the form $w_1w_2\cdots w_k$, where w_1, w_2, \dots, w_k are distinct words of the same length. For example, 001011 is a 3-anti-power, while 001010 is not. In [6], the authors prove that for all positive integers k and r , there is a positive integer $N(k, r)$ such that all words of length at least $N(k, r)$ contain a factor that is either a k -power or an r -anti-power. They also define

E-mail address: cdefant@ufl.edu.

$AP(x, k)$ to be the set of all positive integers m such that the prefix of length km of the word x is a k -anti-power. We will consider this set when $x = \mathbf{t}$ is the Thue-Morse word. It turns out that $AP(\mathbf{t}, k)$ is nonempty for all positive integers k [6, Corollary 6]. It is not difficult to show that if k and m are positive integers, then $m \in AP(\mathbf{t}, k)$ if and only if $2m \in AP(\mathbf{t}, k)$. Therefore, the only interesting elements of $AP(\mathbf{t}, k)$ are those that are odd. For this reason, we make the following definition.

Definition 1.2. Let $\mathcal{F}(k)$ denote the set of odd positive integers m such that the prefix of \mathbf{t} of length km is a k -anti-power. Let $\gamma(k) = \min(\mathcal{F}(k))$ and $\Gamma(k) = \sup((2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k))$.

Remark 1.1. It is immediate from Definition 1.2 that $\mathcal{F}(1) \supseteq \mathcal{F}(2) \supseteq \mathcal{F}(3) \supseteq \dots$. Therefore, $\gamma(1) \leq \gamma(2) \leq \gamma(3) \leq \dots$ and $\Gamma(1) \leq \Gamma(2) \leq \Gamma(3) \leq \dots$.

For convenience, we make the following definition.

Definition 1.3. If m is a positive integer, let $\mathfrak{R}(m)$ denote the smallest positive integer k such that the prefix of \mathbf{t} of length km is not a k -anti-power.

If $k \geq 3$, then $(2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$ is nonempty because it contains the number 3 (the prefix of \mathbf{t} of length 9 is 011010011, which is not a 3-anti-power). We will show (Theorem 3.1) that $(2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$ is finite so that $\Gamma(k)$ is a positive integer for each $k \geq 3$. For example, $(2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(6) = \{1, 3, 9\}$. This means that $AP(\mathbf{t}, 6)$ is the set of all positive integers of the form $2^\ell m$, where ℓ is a nonnegative integer and m is an odd integer that is not 1, 3, or 9.

Fici et al. [6] give the first few values of the sequence $\gamma(k)$ and speculate that the sequence grows linearly in k . We will prove that this is indeed the case. In fact, it is the aim of this paper to prove the following:

$$\begin{aligned} & \bullet \frac{1}{2} \leq \liminf_{k \rightarrow \infty} \frac{\gamma(k)}{k} \leq \frac{9}{10} \\ & \bullet 1 \leq \limsup_{k \rightarrow \infty} \frac{\gamma(k)}{k} \leq \frac{3}{2} \\ & \bullet \liminf_{k \rightarrow \infty} \frac{\Gamma(k)}{k} = \frac{3}{2} \\ & \bullet \limsup_{k \rightarrow \infty} \frac{\Gamma(k)}{k} = 3. \end{aligned}$$

Despite these asymptotic results, there are many open problems arising from consideration of the sets $\mathcal{F}(k)$ (such as the cardinality of $(2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$) that we have not investigated; we discuss some of these problems at the end of the paper.

2. THE THUE-MORSE WORD: BACKGROUND AND NOTATION

Our primary focus is on the Thue-Morse word \mathbf{t} . In this brief section, we discuss some of the basic properties of this word that we will need when proving our asymptotic results.

Let \mathbf{t}_i denote the i^{th} letter of \mathbf{t} so that $\mathbf{t} = \mathbf{t}_1\mathbf{t}_2\mathbf{t}_3\dots$. The number \mathbf{t}_i has the same parity as the number of 1's in the binary expansion of $i - 1$. For any positive integers α, β with $\alpha \leq \beta$, define $\langle \alpha, \beta \rangle = \mathbf{t}_\alpha \mathbf{t}_{\alpha+1} \dots \mathbf{t}_\beta$. In his seminal 1912 paper, Thue proved that \mathbf{t} is overlap-free [7]. This means that if x and y are finite words and x is nonempty, then $xyxyx$ is not a factor of \mathbf{t} .

Equivalently, if a, b, n are positive integers satisfying $a < b \leq a + n$, then $\langle a, a + n \rangle \neq \langle b, b + n \rangle$. Note that this implies that \mathbf{t} is cube-free.

We write $\mathbb{A}^{\leq \omega}$ to denote the set of all words over an alphabet \mathbb{A} . Let \mathcal{W}_1 and \mathcal{W}_2 be sets of words. A *morphism* $f: \mathcal{W}_1 \rightarrow \mathcal{W}_2$ is a function satisfying $f(xy) = f(x)f(y)$ for all words $x, y \in \mathcal{W}_1$. A morphism is uniquely determined by where it sends letters. Let $\mu: \{0, 1\}^{\leq \omega} \rightarrow \{01, 10\}^{\leq \omega}$ denote the morphism defined by $\mu(0) = 01$ and $\mu(1) = 10$. Also, define a morphism $\sigma: \{01, 10\}^{\leq \omega} \rightarrow \{0, 1\}^{\leq \omega}$ by $\sigma(01) = 0$ and $\sigma(10) = 1$ so that $\sigma = \mu^{-1}$. The words \mathbf{t} and $\bar{\mathbf{t}}$ are the unique one-sided infinite words over the alphabet $\{0, 1\}$ that are fixed by μ . Because $\mu(\mathbf{t}) = \mathbf{t}$, we may view \mathbf{t} as a word over the alphabet $\{01, 10\}$. In particular, this means that $\mathbf{t}_{2i-1} \neq \mathbf{t}_{2i}$ for all positive integers i . In addition, if α and β are nonnegative integers with $\alpha < \beta$, then $\langle 2\alpha + 1, 2\beta \rangle \in \{01, 10\}^{\leq \omega}$. Recall the definitions of A_n and B_n from Definition 1.1. Observe that $A_n = \mu^n(0)$ and $B_n = \mu^n(1)$. Because $\mu^n(\mathbf{t}) = \mathbf{t}$, the Thue-Morse word is actually a word over the alphabet $\{A_n, B_n\}$. This leads us to the following simple but useful fact.

Fact 2.1. *For any positive integers n and r , $\langle 2^n r + 1, 2^n(r + 1) \rangle = \mu^n(t_{r+1})$.*

3. ASYMPTOTICS FOR $\Gamma(k)$

In this section, we prove that $\liminf_{k \rightarrow \infty} \Gamma(k)/k = 3/2$ and $\limsup_{k \rightarrow \infty} \Gamma(k)/k = 3$. The following proposition will prove very useful when we do so.

Proposition 3.1. *Let $m \geq 2$ be an integer, and let $\delta(m) = \lceil \log_2(m/3) \rceil$.*

- (i) *If y and v are words such that yvy is a factor of \mathbf{t} and $|y| = m$, then $2^{\delta(m)}$ divides $|yv|$.*
- (ii) *There is a factor of \mathbf{t} of the form yvy such that $|y| = m$ and $2^{\delta(m)+1}$ does not divide $|yv|$.*

Proof. We first prove (ii) by induction on m . If $m = 2$, we may simply set $y = 01$ and $v = 1$. If $m = 3$, we may set $y = 101$ and $v = \varepsilon$ (the empty word). Now, assume $m \geq 4$. First, suppose m is even. By induction, we can find a factor of \mathbf{t} of the form yvy such that $|y| = m/2$ and such that $2^{\delta(m/2)+1}$ does not divide $|yv|$. Note that $\mu(y)\mu(v)\mu(y)$ is a factor of \mathbf{t} and that $2^{\delta(m/2)+2}$ does not divide $2|yv| = |\mu(y)\mu(v)|$. Since $\delta(m/2) + 2 = \delta(m) + 1$, we are done. Now, suppose m is odd. Because $m + 1$ is even, we may use the above argument to find a factor $y'v'y'$ of \mathbf{t} with $|y'| = m + 1$ such that $2^{\delta(m+1)+1}$ does not divide $|y'v'|$. It is easy to show that $\delta(m) = \delta(m + 1)$ because $m > 3$ is odd. This means that $2^{\delta(m)+1}$ does not divide $|y'v'|$. Let a be the last letter of y' , and write $y' = y''a$. Put $v'' = av'$. Then $y''v''y''$ is a factor of \mathbf{t} with $|y''| = m$ and $|y''v''| = |y'v'|$. This completes the inductive step.

We now prove (i) by induction on m . If $m \leq 3$, the proof is trivial because $\delta(2) = \delta(3) = 0$. Therefore, assume $m \geq 4$. Assume that yvy is a factor of \mathbf{t} and $|y| = m$. Let us write $\mathbf{t} = xyvyz$.

Suppose by way of contradiction that $|vy|$ is odd. Then $|xy|$ and $|xyvy|$ have different parities. Write $y = y_1a$, where a is the last letter of y . Either xy or $xyvy$ is an even-length prefix of \mathbf{t} , and is therefore a word in $\{01, 10\}^{\leq \omega}$. It follows that the second-to-last letter of y is \bar{a} , so we may write $y_1 = y_2\bar{a}$. We now observe that one of the words xy_1 and $xyvy_1$ is an even-length prefix of \mathbf{t} , so the same reasoning as before tells us that the second-to-last letter in y_1 is a . Therefore, $y = y_3a\bar{a}a$ for some word y_3 . We can continue in this fashion to see that $a\bar{a}a\bar{a}a$ is a suffix of vy . This is impossible since \mathbf{t} is overlap-free. Hence, $|vy|$ must be even. We now consider four cases corresponding to the possible parities of $|x|$ and m .

Case 1: $|x|$ and $|y| = m$ are both even. We just showed $|vy|$ is even, so all of the words $x, xy, xyv, xyvy$ are even-length prefixes of \mathbf{t} . This means that $x, y, v, z \in \{01, 10\}^{\leq \omega}$, so $\mathbf{t} = \sigma(x)\sigma(y)\sigma(v)\sigma(y)\sigma(z)$. By induction, we see that $2^{\delta(|\sigma(y)|)}$ divides $|\sigma(y)\sigma(v)|$. Because $\delta(|\sigma(y)|) = \delta(m/2) = \delta(m) - 1$ and $|\sigma(y)\sigma(v)| = |yv|/2$, it follows that $2^{\delta(m)}$ divides $|yv|$.

Case 2: $|x|$ is odd and m is even. As in the previous case, $|v|$ must be even. Let a, b, c be the last letters of y, v, x , respectively. Write $y = y_0a$, $v = v_0b$, $x = x_0c$. We have $\mathbf{t} = x_0cy_0av_0by_0az$. Note that $|x_0|$, $|cy_0|$, $|av_0|$, and $|by_0|$ are all even. In particular, cy_0 and by_0 are both in $\{01, 10\}^{\leq \omega}$. As a consequence, $b = c$. Setting $x' = x_0$, $y' = by_0$, $v' = av_0$, $z' = az$, we find that $\mathbf{t} = x'y'v'y'z'$. We are now in the same situation as in the previous case because $|x'|$ is even and $|y'| = m$. Consequently, $2^{\delta(m)}$ divides $|y'v'| = |yv|$.

Case 3: m is odd and $|x|$ is even. Let a be the last letter of y . Both v and z start with the letter \bar{a} , so we may write $v = \bar{a}v_1$ and $z = \bar{a}z_1$. Put $x_1 = x$ and $y_1 = y\bar{a}$. We have $\mathbf{t} = x_1y_1v_1y_1z_1$. Because $|x_1|$ and $|y_1| = m + 1$ are both even, we know from the first case that $2^{\delta(m+1)}$ divides $|y_1v_1| = |yv|$. Now, simply observe that $\delta(m) = \delta(m + 1)$ because $m > 3$ is odd.

Case 4: m and $|x|$ are both odd. Let d be the first letter of y . Both x and v end in the letter \bar{d} , so we may write $x = x_2\bar{d}$ and $v = v_2\bar{d}$. Let $y_2 = \bar{d}y$ and $z_2 = z$. Then $\mathbf{t} = x_2y_2v_2y_2z_2$. Because $|x_2|$ and $|y_2| = m + 1$ are both even, we know that $2^{\delta(m+1)}$ divides $|y_2v_2| = |yv|$. Again, $\delta(m) = \delta(m + 1)$. \square

Corollary 3.1. *Let m be a positive integer, and let $\delta(m) = \lceil \log_2(m/3) \rceil$. If $k \geq 3$ and $m \in (2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$, then $k - 1 \geq 2^{\delta(m)}$.*

Proof. There exist integers n_1 and n_2 with $0 \leq n_1 < n_2 \leq k - 1$ such that $\langle n_1m + 1, (n_1 + 1)m \rangle = \langle n_2m + 1, (n_2 + 1)m \rangle$. Let $y = \langle n_1m + 1, (n_1 + 1)m \rangle$ and $v = \langle (n_1 + 1)m + 1, n_2m \rangle$. The word yvy is a factor of \mathbf{t} , and $|y| = m$. According to Proposition 3.1, $2^{\delta(m)}$ divides $|yv| = (n_2 - n_1)m$, where $\delta(m) = \lceil \log_2(m/3) \rceil$. Since m is odd, $2^{\delta(m)}$ divides $n_2 - n_1$. This shows that $k - 1 \geq n_2 \geq n_2 - n_1 \geq 2^{\delta(m)}$. \square

The following lemma is somewhat technical, but it will be useful for constructing specific pairs of identical factors of the Thue-Morse word. These specific pairs of factors will provide us with odd positive integers m for which $\mathfrak{R}(m)$ is relatively small. We will then make use of the fact, which follows immediately from Definitions 1.2 and 1.3, that $\Gamma(k) \geq m$ whenever $k \geq \mathfrak{R}(m)$.

Lemma 3.1. *Suppose r, m, ℓ, h, p, q are nonnegative integers satisfying the following conditions:*

- $h < 2^{\ell-2}$
- $rm = p \cdot 2^{\ell+1} + 2^{\ell-1} + h$
- $(r + 1)m \leq p \cdot 2^{\ell+1} + 5 \cdot 2^{\ell-2}$
- $(r + 2^{\ell-2})m = q \cdot 2^{\ell+1} + 3 \cdot 2^{\ell-2} + h$
- $\mathbf{t}_{p+1} \neq \mathbf{t}_{q+1}$.

Then $\langle rm + 1, (r + 1)m \rangle = \langle (r + 2^{\ell-2})m + 1, (r + 2^{\ell-2} + 1)m \rangle$, and $\mathfrak{R}(m) \leq r + 2^{\ell-2} + 1$.

Proof. Let $u = \langle rm + 1, (r + 1)m \rangle$ and $v = \langle (r + 2^{\ell-2})m + 1, (r + 2^{\ell-2} + 1)m \rangle$. Let us assume $\mathbf{t}_{p+1} = 0$; a similar argument holds if we assume instead that $\mathbf{t}_{p+1} = 1$. According to Fact 2.1,

$$\langle p \cdot 2^{\ell+1} + 1, (p + 1)2^{\ell+1} \rangle = A_{\ell+1} = A_{\ell-2}B_{\ell-2}B_{\ell-2}A_{\ell-2}B_{\ell-2}A_{\ell-2}A_{\ell-2}B_{\ell-2}.$$

$A_{\ell+1}$								$B_{\ell+1}$							
$A_{\ell-2}$	$B_{\ell-2}$	$B_{\ell-2}$	$A_{\ell-2}$	$B_{\ell-2}$	$A_{\ell-2}$	$A_{\ell-2}$	$B_{\ell-2}$	$B_{\ell-2}$	$A_{\ell-2}$	$A_{\ell-2}$	$B_{\ell-2}$	$A_{\ell-2}$	$B_{\ell-2}$	$B_{\ell-2}$	$A_{\ell-2}$
x		u		y								x'	v		y'

FIGURE 1. An illustration of the proof of Lemma 3.1.

We may now use the first three conditions to see that $B_{\ell-2}A_{\ell-2}B_{\ell-2} = xuy$ for some words x and y such that $|x| = h$ and $|y| = p \cdot 2^{\ell+1} + 5 \cdot 2^{\ell-2} - (r+1)m$ (see Figure 1).

We know from the last condition that $\mathbf{t}_{q+1} = 1$, so

$$\langle q \cdot 2^{\ell+1} + 1, (q+1)2^{\ell+1} \rangle = B_{\ell+1} = B_{\ell-2}A_{\ell-2}A_{\ell-2}B_{\ell-2}A_{\ell-2}B_{\ell-2}B_{\ell-2}A_{\ell-2}.$$

The fourth condition tells us that $B_{\ell-2}A_{\ell-2}B_{\ell-2} = x'vy'$ for some words x' and y' with $|x'| = h$. We have shown that $xuy = x'vy'$, where $|x| = |x'|$ and $|u| = |v|$. Hence, $u = v$. It follows that the prefix of \mathbf{t} of length $(r + 2^{\ell-2} + 1)m$ is not a $(r + 2^{\ell-2} + 1)$ -anti-power, so $\mathfrak{R}(m) \leq r + 2^{\ell-2} + 1$ by definition. \square

We may now use Lemma 3.1 and Proposition 3.1 to prove that $\limsup_{k \rightarrow \infty} \Gamma(k)/k = 3$. Recall that if $k \geq 3$, then $\Gamma(k) \geq 3$ because $3 \in (2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$. A particular consequence of the following theorem is that $(2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$ is finite. It follows that if $k \geq 3$, then $\Gamma(k)$ is an odd positive integer.

Theorem 3.1. *Let $\Gamma(k)$ be as in Definition 1.2. For all integers $k \geq 3$, we have $\Gamma(k) \leq 3k - 4$. Furthermore, $\limsup_{k \rightarrow \infty} \frac{\Gamma(k)}{k} = 3$.*

Proof. Fix $k \geq 3$, and let $m \in (2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$. If $m \leq 5$, then $m \leq 3k - 4$ as desired, so assume $m \geq 7$. By Corollary 3.1, $k - 1 \geq 2^{\delta(m)}$, where $\delta(m) = \lceil \log_2(m/3) \rceil$. Since $m \geq 7$ is odd, $\delta(m) > \log_2(m/3)$. This shows that $k - 1 \geq 2^{\delta(m)} > m/3$, so $m \leq 3k - 4$. Consequently, $\Gamma(k) \leq 3k - 4$.

We now show that $\limsup_{k \rightarrow \infty} \frac{\Gamma(k)}{k} = 3$. For each positive integer α , let $k_\alpha = 2^{2\alpha} + 2^\alpha + 2$. Let us fix an integer $\alpha \geq 3$ and set $r = 2^\alpha + 1$, $m = 3 \cdot 2^{2\alpha} - 2^\alpha + 1$, $\ell = 2\alpha + 2$, $h = 1$, $p = 3 \cdot 2^{\alpha-3}$, and $q = 3 \cdot 2^{2\alpha-3} + 2^{\alpha-2}$. One may easily verify that these values of r, m, ℓ, h, p , and q satisfy the first four of the five conditions listed in Lemma 3.1. Recall that the parity of \mathbf{t}_i is the same as the parity of the number of 1's in the binary expansion of $i - 1$. The binary expansion of p has exactly two 1's, and the binary expansion of q has exactly three 1's. Therefore, $\mathbf{t}_{p+1} = 0 \neq 1 = \mathbf{t}_{q+1}$. This shows that all of the conditions in Lemma 3.1 are satisfied, so $\mathfrak{R}(m) \leq r + 2^{\ell-2} + 1 = k_\alpha$. The prefix of \mathbf{t} of length $k_\alpha m$ is not a k_α -anti-power, so $\Gamma(k_\alpha) \geq m = 3 \cdot 2^{2\alpha} - 2^\alpha + 1$. For each $\alpha \geq 3$,

$$\frac{\Gamma(k_\alpha)}{k_\alpha} \geq \frac{3 \cdot 2^{2\alpha} - 2^\alpha + 1}{2^{2\alpha} + 2^\alpha + 2}. \quad \square$$

In the preceding proof, we found an increasing sequence of positive integers $(k_\alpha)_{\alpha \geq 3}$ with the property that $\Gamma(k_\alpha)/k_\alpha \rightarrow 3$ as $\alpha \rightarrow \infty$. It will be useful to have two other sequences with similar properties. This is the content of the following lemma.

Lemma 3.2. *For integers $\alpha \geq 3$, $\beta \geq 9$, and $\rho \geq 4$, define*

$$k_\alpha = 2^{2\alpha} + 2^\alpha + 2, \quad K_\beta = 2^{2\beta+1} + 3 \cdot 2^{\beta+3} + 49, \quad \text{and} \quad \kappa_\rho = 2^\rho + 2.$$

We have

$$\Gamma(k_\alpha) \geq 3 \cdot 2^{2\alpha} - 2^\alpha + 1, \quad \Gamma(K_\beta) \geq 3 \cdot 2^{2\beta+1} - 2^{\beta-1} + 1, \quad \text{and} \quad \Gamma(\kappa_\rho) \geq 5 \cdot 2^{\rho-1} - 8\chi(\rho) + 1,$$

$$\text{where } \chi(\rho) = \begin{cases} 1, & \text{if } \rho \equiv 0 \pmod{2}; \\ 2, & \text{if } \rho \equiv 1 \pmod{2}. \end{cases}$$

Proof. We already derived the lower bound for $\Gamma(k_\alpha)$ in the proof of Theorem 3.1. To prove the lower bound for $\Gamma(K_\beta)$, put $r = 3 \cdot 2^{\beta+3} + 48$, $m = 3 \cdot 2^{2\beta+1} - 2^{\beta-1} + 1$, $\ell = 2\beta + 3$, $h = 48$, $p = 9 \cdot 2^\beta + 17$, and $q = 3 \cdot 2^{2\beta-2} + 143 \cdot 2^{\beta-4} + 17$. Straightforward calculations show that these choices of r, m, ℓ, h, p , and q satisfy the first four conditions of Lemma 3.1. The binary expansion of p has exactly four 1's while that of q has exactly nine 1's (it is here that we require $\beta \geq 9$). It follows that $\mathbf{t}_{p+1} = 0 \neq 1 = \mathbf{t}_{q+1}$, so the final condition in Lemma 3.1 is also satisfied. The lemma tells us that $\mathfrak{R}(m) \leq r + 2^{\ell-2} + 1 = K_\beta$, so the prefix of \mathbf{t} of length $K_\beta m$ is not a K_β -anti-power. Hence, $\Gamma(K_\beta) \geq m = 3 \cdot 2^{2\beta+1} - 2^{\beta-1} + 1$.

To prove the lower bound for κ_ρ , we again invoke Lemma 3.1. Let $r' = 1$, $m' = 5 \cdot 2^{\rho-1} - 8\chi(\rho) + 1$, $\ell' = \rho + 2$, $h' = 2^{\rho-1} - 8\chi(\rho) + 1$, $p' = 0$, and $q' = 5 \cdot 2^{\rho-4} - \chi(\rho)$. These choices satisfy the first four conditions in Lemma 3.1. The binary expansion of q' has an odd number of 1's, so $\mathbf{t}_{p'+1} = \mathbf{t}_1 = 0 \neq 1 = \mathbf{t}_{q'+1}$. We now know that $\mathfrak{R}(m') \leq r' + 2^{\ell'-2} + 1 = \kappa_\rho$, so $\Gamma(\kappa_\rho) \geq m' = 5 \cdot 2^{\rho-1} - 8\chi(\rho) + 1$. \square

We now use the sequences $(k_\alpha)_{\alpha \geq 3}$, $(K_\beta)_{\beta \geq 9}$, and $(\kappa_\rho)_{\rho \geq 4}$ to prove that $\liminf_{k \rightarrow \infty} (\Gamma(k)/k) = 3/2$.

Theorem 3.2. *Let $\Gamma(k)$ be as in Definition 1.2. We have $\liminf_{k \rightarrow \infty} \frac{\Gamma(k)}{k} = \frac{3}{2}$.*

Proof. Let $k \geq 3$ be a positive integer, and let $m = \Gamma(k)$. Put $\delta(m) = \lceil \log_2(m/3) \rceil$. Corollary 3.1 tells us that $k - 1 \geq 2^{\delta(m)}$. Suppose k is a power of 2, say $k = 2^\lambda$. Then the inequality $k - 1 \geq 2^{\delta(m)}$ forces $\delta(m) \leq \lambda - 1$. Thus, $m \leq 3 \cdot 2^{\lambda-1} = \frac{3}{2}k$. This shows that $\frac{\Gamma(k)}{k} \leq \frac{3}{2}$ whenever k is a power of 2, so $\liminf_{k \rightarrow \infty} \frac{\Gamma(k)}{k} \leq \frac{3}{2}$.

To prove the reverse inequality, we will make use of Lemma 3.2. Recall the definitions of k_α , K_β , κ_ρ , and $\chi(\rho)$ from that lemma. Fix $k \geq \kappa_{18}$, and put $m = \Gamma(k)$. Because $k \geq \kappa_{18}$, we may use Lemma 3.2 and the fact that Γ is nondecreasing (see Remark 1.1) to see that $m = \Gamma(k) \geq \Gamma(\kappa_{18}) \geq 5 \cdot 2^{17} - 7$. Let $\ell = \lceil \log_2 m \rceil$ so that $2^{\ell-1} < m < 2^\ell$. Note that $\ell \geq 20$. Let us first assume that $3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2} < m < 2^\ell$. Lemma 3.2 tells us that $\Gamma(\kappa_{\ell-1}) \geq 5 \cdot 2^{\ell-2} - 8\chi(\ell-1) + 1$. We also know that $5 \cdot 2^{\ell-2} - 8\chi(\ell-1) + 1 > m$, so $\Gamma(\kappa_{\ell-1}) > m$. Because Γ is nondecreasing, $\kappa_{\ell-1} > k$. Thus,

$$(1) \quad \frac{\Gamma(k)}{k} > \frac{3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2}}{\kappa_{\ell-1}} = \frac{3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2}}{2^{\ell-1} + 2}$$

if $3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2} < m < 2^\ell$.

Next, assume $2^{\ell-1} < m \leq 3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2}$ and ℓ is even. According to Lemma 3.2, $\Gamma(k_{(\ell-2)/2}) \geq 3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2} + 1 > m$. Because Γ is nondecreasing, $k < k_{(\ell-2)/2}$. Therefore,

$$(2) \quad \frac{\Gamma(k)}{k} > \frac{2^{\ell-1}}{k_{(\ell-2)/2}} = \frac{2^{\ell-1}}{2^{\ell-2} + 2^{(\ell-2)/2} + 2}.$$

Finally, suppose $2^{\ell-1} < m \leq 3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2}$ and ℓ is odd. Lemma 3.2 states that $\Gamma(K_{(\ell-3)/2}) \geq 3 \cdot 2^{\ell-2} - 2^{(\ell-5)/2} + 1 > m$. We know that $k < K_{(\ell-3)/2}$ because Γ is nondecreasing. As a consequence,

$$(3) \quad \frac{\Gamma(k)}{k} > \frac{2^{\ell-1}}{K_{(\ell-3)/2}} = \frac{2^{\ell-1}}{2^{\ell-2} + 3 \cdot 2^{(\ell+3)/2} + 49}.$$

The inequalities in (1), (2), and (3) show that in all cases, $\frac{\Gamma(k)}{k} > \frac{3 \cdot 2^{\ell-2} - 2^{(\ell-2)/2}}{2^{\ell-1} + 2}$. Because $\ell \rightarrow \infty$ as $k \rightarrow \infty$ ($\Gamma(k)$ cannot be bounded since we have just shown $\Gamma(k)/k$ is bounded away from 0), we find that $\liminf_{k \rightarrow \infty} \Gamma(k)/k \geq 3/2$. \square

4. ASYMPTOTICS FOR $\gamma(k)$

Having demonstrated that $\liminf_{k \rightarrow \infty} (\Gamma(k)/k) = 3/2$ and $\limsup_{k \rightarrow \infty} (\Gamma(k)/k) = 3$, we turn our attention to $\gamma(k)$. To begin the analysis, we prove some lemmas that culminate in an upper bound for $\mathfrak{R}(m)$ for any odd positive integer m . It will be useful to keep in mind that if j is a nonnegative integer, then $\mathbf{t}_{2j} \neq \mathbf{t}_{2j+1} = \mathbf{t}_{j+1}$ and $\mathbf{t}_{4j+2} = \mathbf{t}_{4j+3}$.

Lemma 4.1. *Let m be an odd positive integer, and let $\ell = \lceil \log_2 m \rceil$. If $\mathfrak{R}(m) > 2^\ell + 1$, then $\mathbf{t}_{m+1}\mathbf{t}_{m+2} = 11$ and $\mathbf{t}_{2m+1}\mathbf{t}_{2m+2} = 10$.*

Proof. Let $w_0 = \langle 1, m \rangle$, $w_1 = \langle 2^{\ell-1}m + 1, (2^{\ell-1} + 1)m \rangle$, and $w_2 = \langle 2^\ell m + 1, (2^\ell + 1)m \rangle$. The words w_0, w_1, w_2 must be distinct because $\mathfrak{R}(m) > 2^\ell + 1$. For each $n \in \{0, 1, 2\}$, w_n is a prefix of $\langle nm2^{\ell-1} + 1, (nm + 2)2^{\ell-1} \rangle = \mu^{\ell-1}(\mathbf{t}_{nm+1}\mathbf{t}_{nm+2})$. It follows that $\mathbf{t}_1\mathbf{t}_2$, $\mathbf{t}_{m+1}\mathbf{t}_{m+2}$, and $\mathbf{t}_{2m+1}\mathbf{t}_{2m+2}$ are distinct. Since $\mathbf{t}_1\mathbf{t}_2 = 01$ and $\mathbf{t}_{2m+1} \neq \mathbf{t}_{2m+2}$, we must have $\mathbf{t}_{2m+1}\mathbf{t}_{2m+2} = 10$. Now, $\mathbf{t}_{2m+1}\mathbf{t}_{2m+2} = \mu(\mathbf{t}_{m+1})$, so $\mathbf{t}_{m+1} = 1$. This forces $\mathbf{t}_{m+1}\mathbf{t}_{m+2} = 11$. \square

Lemma 4.2. *Let $m \geq 3$ be an odd integer, and let $\ell = \lceil \log_2 m \rceil$. Suppose there is a positive integer j such that $\mathbf{t}_j\mathbf{t}_{j+1} = \mathbf{t}_{m+j}\mathbf{t}_{m+j+1}$. Then $\mathfrak{R}(m) < \left(1 + \frac{j+1}{m}\right) 2^\ell$.*

Proof. First, observe that

$$(4) \quad \langle 2^\ell(j-1) + 1, 2^\ell(j+1) \rangle = \mu^\ell(\mathbf{t}_j\mathbf{t}_{j+1}) = \mu^\ell(\mathbf{t}_{m+j}\mathbf{t}_{m+j+1}) = \langle 2^\ell(m+j-1) + 1, 2^\ell(m+j+1) \rangle.$$

Because $|\langle 2^\ell(j-1) + 1, 2^\ell(j+1) \rangle| = 2^{\ell+1} > 2m$, there is a nonnegative integer r such that

$$(5) \quad \langle 2^\ell(j-1) + 1, 2^\ell(j+1) \rangle = w \langle rm + 1, (r+1)m \rangle z$$

for some nonempty words w and z . Note that $r+1 < \frac{2^\ell(j+1)}{m}$. It follows from (5) that

$$2^\ell(m+j-1) + 1 < 2^\ell m + rm + 1 < 2^\ell m + (r+1)m < 2^\ell(m+j+1),$$

so

$$\langle 2^\ell(m+j-1) + 1, 2^\ell(m+j+1) \rangle = w' \langle (2^\ell + r)m + 1, (2^\ell + r + 1)m \rangle z'$$

for some nonempty words w' and z' . Note that $|w'| = (2^\ell + r)m - 2^\ell(m + j - 1) = rm - 2^\ell(j - 1) = |w|$. Combining this fact with (4), we find that

$$\langle rm + 1, (r + 1)m \rangle = \langle (2^\ell + r)m + 1, (2^\ell + r + 1)m \rangle.$$

Consequently,

$$\mathfrak{K}(m) \leq 2^\ell + r + 1 < 2^\ell + \frac{2^\ell(j + 1)}{m}.$$

□

Lemma 4.3. *Let m be an odd positive integer with $m \not\equiv 1 \pmod{8}$, and let $\ell = \lceil \log_2 m \rceil$. We have $\mathfrak{K}(m) < (1 + \frac{37}{m}) 2^\ell$.*

Proof. Suppose instead that $\mathfrak{K}(m) \geq (1 + \frac{37}{m}) 2^\ell$. Let us assume for the moment that $m \not\equiv 29 \pmod{32}$. We will obtain a contradiction to Lemma 4.2 by exhibiting a positive integer $j \leq 36$ such that $\mathbf{t}_j \mathbf{t}_{j+1} = \mathbf{t}_{m+j} \mathbf{t}_{m+j+1}$. Because $\mathfrak{K}(m) > 2^\ell + 1$, Lemma 4.1 tells us that $\mathbf{t}_{m+1} \mathbf{t}_{m+2} = 11$ and $\mathbf{t}_{2m+1} \mathbf{t}_{2m+2} = 10$.

First, assume $m \equiv 3 \pmod{4}$. We have $\langle m + 2, m + 5 \rangle = \mu^2(\mathbf{t}_{(m+5)/4})$, so either $\langle m + 2, m + 5 \rangle = 0110$ or $\langle m + 2, m + 5 \rangle = 1001$. Since $\mathbf{t}_{m+2} = 1$, we must have $\langle m + 2, m + 5 \rangle = 1001$. This shows that $\mathbf{t}_{m+4} \mathbf{t}_{m+5} = 01 = \mathbf{t}_4 \mathbf{t}_5$, so we may set $j = 4$.

Next, assume $m \equiv 5 \pmod{8}$. Let $x01^s01$ be the binary expansion of m , where x is some (possibly empty) string of 0's and 1's. As $m \equiv 5 \pmod{8}$ and $m \not\equiv 29 \pmod{32}$, we must have $1 \leq s \leq 2$. Because $\mathbf{t}_{m+1} = 1$, the number of 1's in the binary expansion of m is odd. This means that the parity of the number of 1's in x is the same as the parity of s .

Suppose $s = 1$. The binary expansion of $m + 3$ is the string $x1000$, which contains an even number of 1's. As a consequence, $\mathbf{t}_{m+4} = 0$. The binary expansion of $m + 4$ is $x1001$, so $\mathbf{t}_{m+5} = 1$. This shows that $\mathbf{t}_{m+4} \mathbf{t}_{m+5} = 01 = \mathbf{t}_4 \mathbf{t}_5$, so we may set $j = 4$.

Suppose that $s = 2$ and that x ends in a 0, say $x = y0$. Note that y contains an even number of 1's. The binary expansions of $m + 19$ and $m + 20$ are $y100000$ and $y100001$, respectively, so $\mathbf{t}_{m+20} \mathbf{t}_{m+21} = 10 = \mathbf{t}_{20} \mathbf{t}_{21}$. We may set $j = 20$ in this case.

Assume now that $s = 2$ and that x ends in a 1. Let us write $x = x'01^{s'}$, where x' is a (possibly empty) binary string. For this last step, we may need to add additional 0's to the beginning of x . Doing so does not raise any issues because it does not change the number of 1's in x . The binary expansion of m is $x'01^{s'}01101$. Note that the parity of the number of 1's in x' is the same as the parity of s' . The binary expansions of $m + 19$ and $m + 35$ are $x'10^{s'+5}$ and $x'10^{s'}10000$, respectively. If s' is even, then we may put $j = 20$ because $\mathbf{t}_{m+20} \mathbf{t}_{m+21} = 10 = \mathbf{t}_{20} \mathbf{t}_{21}$. If s' is odd, then we may set $j = 36$ because $\mathbf{t}_{m+36} \mathbf{t}_{m+37} = 10 = \mathbf{t}_{36} \mathbf{t}_{37}$.

We now handle the case in which $m \equiv 29 \pmod{32}$. Say $m = 32n - 3$. Let b be the number of 1's in the binary expansion of n . The binary expansion of $m + 17 = 32n + 14$ has $b + 3$ 1's. Similarly, the binary expansions of $m + 18$, $m + 19$, $2m + 17$, $2m + 18$, and $2m + 19$ have $b + 4$, $b + 1$, $b + 3$, $b + 2$, and $b + 3$ 1's, respectively. This means that $\mathbf{t}_{m+18} \mathbf{t}_{m+19} \mathbf{t}_{m+20} = \mathbf{t}_{2m+18} \mathbf{t}_{2m+19} \mathbf{t}_{2m+20}$. Therefore,

$$\begin{aligned} \langle (m + 17)2^{\ell-1} + 1, (m + 20)2^{\ell-1} \rangle &= \mu^{\ell-1}(\mathbf{t}_{m+18} \mathbf{t}_{m+19} \mathbf{t}_{m+20}) \\ (6) \quad &= \mu^{\ell-1}(\mathbf{t}_{2m+18} \mathbf{t}_{2m+19} \mathbf{t}_{2m+20}) = \langle (2m + 17)2^{\ell-1} + 1, (2m + 20)2^{\ell-1} \rangle. \end{aligned}$$

We have $\bigcup_{r=9}^{17} \left(\frac{17}{2r}, \frac{10}{r+1} \right) = \left(\frac{1}{2}, 1 \right)$, so there exists some $r \in \{9, 10, \dots, 17\}$ such that $\frac{17}{2r} < \frac{m}{2^\ell} < \frac{10}{r+1}$. Equivalently, $17 \cdot 2^{\ell-1} < rm < (r+1)m < 20 \cdot 2^{\ell-1}$. It follows that there are nonempty words w and z such that $\langle (m+17)2^{\ell-1} + 1, (m+20)2^{\ell-1} \rangle = w \langle (r+2^{\ell-1})m + 1, (r+2^{\ell-1}+1)m \rangle z$. Similarly, there are nonempty words w' and z' such that $\langle (2m+17)2^{\ell-1} + 1, (2m+20)2^{\ell-1} \rangle = w' \langle (r+2^\ell)m + 1, (r+2^\ell+1)m \rangle z'$. Note that $|w| = rm - 17 \cdot 2^{\ell-1} = |w'|$. Invoking (6) yields $\langle (r+2^{\ell-1})m + 1, (r+2^{\ell-1}+1)m \rangle = \langle (r+2^\ell)m + 1, (r+2^\ell+1)m \rangle$. This shows that $\mathfrak{K}(m) \leq r+2^\ell+1 \leq 2^\ell+18$, securing our final contradiction to the assumption that $\mathfrak{K}(m) \geq (1 + \frac{37}{m})2^\ell$. \square

Lemma 4.4. *Let m be an odd positive integer, and let $\ell = \lceil \log_2 m \rceil$. Suppose $m = 2^L h + 1$, where L and h are integers with $L \geq 3$ and h odd. We have $\mathfrak{K}(m) < \left(1 + \frac{2^{L+1} + 4}{m}\right) 2^\ell$.*

Proof. Suppose instead that $\mathfrak{K}(m) \geq \left(1 + \frac{2^{L+1} + 4}{m}\right) 2^\ell$. We will obtain a contradiction to Lemma 4.2 by finding a positive integer $j \leq 2^{L+1} + 3$ satisfying $\mathbf{t}_j \mathbf{t}_{j+1} = \mathbf{t}_{m+j} \mathbf{t}_{m+j+1}$. Let $x01^s0^{L-1}1$ be the binary expansion of m , and note that $s \geq 1$. Let N be the number of 1's in x . The binary expansions of $m + 2^L + 2$, $m + 2^L + 3$, $m + 2^{L+1} + 2$, and $m + 2^{L+1} + 3$ are $x10^{s+L-2}11$, $x10^{s+L-3}100$, $x10^{s-1}10^{L-2}11$, and $x10^{s-1}10^{L-3}100$. This shows that $\mathbf{t}_{m+2^L+3} \mathbf{t}_{m+2^L+4} = 10$ if N is even and $\mathbf{t}_{m+2^{L+1}+3} \mathbf{t}_{m+2^{L+1}+4} = 10$ if N is odd. Observe that $\mathbf{t}_{2^L+3} \mathbf{t}_{2^L+4} = \mathbf{t}_{2^{L+1}+3} \mathbf{t}_{2^{L+1}+4} = 10$. Therefore, we may put $j = 2^L + 3$ if N is even and $j = 2^{L+1} + 3$ if N is odd. \square

Lemma 4.5. *Let m be an odd positive integer, and let $\ell = \lceil \log_2 m \rceil$. Assume $m = 2^L h + 1$ for some integers L and h with $L \geq 3$ and h odd. If n is an integer such that $2 \leq n \leq 2^{L-1}$, $\mathbf{t}_{m-n} = \mathbf{t}_{m-n+1}$, and $m \leq \left(1 - \frac{1}{2n+2}\right) 2^\ell$, then $\mathfrak{K}(m) \leq 2^\ell - n$.*

Proof. Let y and z be the binary expansions of $2^{L-1} - n$ and $2^{L-1} - n + 1$, respectively. If necessary, let the strings y and z begin with additional 0's so that $|y| = |z| = L - 1$. Let $x10^L$ be the binary expansion of $m - 1$. The binary expansions of $m - 2n - 1$ and $2m - 2n - 1$ are $x0y0$ and $x01y1$, respectively. The quantities of 1's in these strings are of the same parity, so $\mathbf{t}_{m-2n} = \mathbf{t}_{2m-2n}$. Similarly, $\mathbf{t}_{m-2n+2} = \mathbf{t}_{2m-2n+2}$ because the binary expansions of $m - 2n + 1$ and $2m - 2n + 1$ are $x0z0$ and $x01z1$, respectively. Let $a = \mathbf{t}_{m-n}$. Because $\mathbf{t}_{m-n} = \mathbf{t}_{m-n+1}$ by hypothesis, we have $\mathbf{t}_{2m-2n} = \mathbf{t}_{2m-2n+2} = \bar{a}$. Therefore, $\mathbf{t}_{m-2n} = \mathbf{t}_{m-2n+2} = \bar{a}$. The word \mathbf{t} is cube-free, so $\mathbf{t}_{m-2n} \mathbf{t}_{m-2n+1} \mathbf{t}_{m-2n+2} = \bar{a} a \bar{a} = \mathbf{t}_{2m-2n} \mathbf{t}_{2m-2n+1} \mathbf{t}_{2m-2n+2}$. Hence,

$$\begin{aligned} \langle (m-2n-1)2^{\ell-1} + 1, (m-2n+2)2^{\ell-1} \rangle &= \mu^{\ell-1}(\mathbf{t}_{m-2n} \mathbf{t}_{m-2n+1} \mathbf{t}_{m-2n+2}) \\ (7) \quad &= \mu^{\ell-1}(\mathbf{t}_{2m-2n} \mathbf{t}_{2m-2n+1} \mathbf{t}_{2m-2n+2}) = \langle (2m-2n-1)2^{\ell-1} + 1, (2m-2n+2)2^{\ell-1} \rangle. \end{aligned}$$

Now, $m \in \left(2^{\ell-1}, \left(1 - \frac{1}{2n+2}\right) 2^\ell\right] \subseteq \bigcup_{r=n}^{2n-1} \left[\frac{2n-2}{r} 2^{\ell-1}, \frac{2n+1}{r+1} 2^{\ell-1}\right]$, so there is some $r \in \{n, n+1, \dots, 2n-1\}$ such that $\frac{2n-2}{r} 2^{\ell-1} \leq m \leq \frac{2n+1}{r+1} 2^{\ell-1}$. Equivalently, $(m-2n-1)2^{\ell-1} \leq (2^{\ell-1} - r - 1)m < (2^{\ell-1} - r)m \leq (m-2n+2)2^{\ell-1}$. We find that

$$\langle (m-2n-1)2^{\ell-1} + 1, (m-2n+2)2^{\ell-1} \rangle = w \langle (2^{\ell-1} - r - 1)m + 1, (2^{\ell-1} - r)m \rangle z$$

and

$$\langle (2m-2n-1)2^{\ell-1} + 1, (2m-2n+2)2^{\ell-1} \rangle = w' \langle (2^\ell - r - 1)m + 1, (2^\ell - r)m \rangle z'$$

for some words w, w', z, z' . Because $|w| = (2n+1)2^{\ell-1} - (r+1)m = |w'|$, we may use (7) to deduce that

$$\langle (2^{\ell-1} - r - 1)m + 1, (2^{\ell-1} - r)m \rangle = \langle (2^\ell - r - 1)m + 1, (2^\ell - r)m \rangle.$$

This shows that $\mathfrak{K}(m) \leq 2^\ell - r \leq 2^\ell - n$ as desired. \square

Lemma 4.6. *If m is an odd positive integer and $\ell = \lceil \log_2 m \rceil$, then $\mathfrak{K}(m) < 2^\ell + 2^{(\ell+5)/2} + 10$.*

Proof. We will assume that $m \geq 65$ (so $\ell \geq 7$). One may easily use a computer to check that the desired result holds when $m < 65$.

If $m \not\equiv 1 \pmod{8}$, then Lemma 4.3 tells us that

$$\mathfrak{K}(m) < \left(1 + \frac{37}{m}\right) 2^\ell < 2^\ell + 74 \leq 2^\ell + 2^{(\ell+5)/2} + 10.$$

Suppose that $m \equiv 1 \pmod{8}$, and let $m = 2^L h + 1$, where $L \geq 3$ and h is odd. First, assume $m > \left(1 - \frac{1}{2^L - 4}\right) 2^\ell$. Because $2^L |2^\ell - m + 1|$ and $2^\ell - m + 1 > 0$, we have $2^L \leq 2^\ell - m + 1 < \frac{2^\ell}{2^L - 4} + 1$. This implies that $2^{2L} - 4 \cdot 2^L < 2^\ell + 2^L - 4$, so $2^L < 2^{\ell-L} + 5 - 4 \cdot 2^{-L} < 2^{\ell-L+2}$. Hence, $L \leq \frac{\ell+1}{2}$. By Lemma 4.4,

$$\mathfrak{K}(m) < \left(1 + \frac{2^{L+1} + 4}{m}\right) 2^\ell < 2^\ell + 2^{L+2} + 8 < 2^\ell + 2^{(\ell+5)/2} + 10.$$

Next, assume $m \leq \left(1 - \frac{1}{2^L - 4}\right) 2^\ell$ and $L \geq 4$. Let n be the largest integer satisfying $m - n \equiv 2 \pmod{4}$ and $n \leq 2^{L-1}$. Note that $m \leq \left(1 - \frac{1}{2n+2}\right) 2^\ell$ because $n \geq 2^{L-1} - 3$. As $m - n \equiv 2 \pmod{4}$, we have $\mathbf{t}_{m-n} = \mathbf{t}_{m-n+1}$. We have shown that n satisfies the criteria specified in Lemma 4.5, so $\mathfrak{K}(m) \leq 2^\ell - n < 2^\ell + 2^{(\ell+5)/2} + 10$.

Finally, if $L = 3$, then Lemma 4.4 tells us that

$$\mathfrak{K}(m) < \left(1 + \frac{20}{m}\right) 2^\ell < 2^\ell + 40 < 2^\ell + 2^{(\ell+5)/2} + 10. \quad \square$$

At last, we are in a position to prove lower bounds for $\liminf_{k \rightarrow \infty} (\gamma(k)/k)$ and $\limsup_{k \rightarrow \infty} (\gamma(k)/k)$.

Theorem 4.1. *Let $\gamma(k)$ be as in Definition 1.2. We have*

$$\liminf_{k \rightarrow \infty} \frac{\gamma(k)}{k} \geq \frac{1}{2} \quad \text{and} \quad \limsup_{k \rightarrow \infty} \frac{\gamma(k)}{k} \geq 1.$$

Proof. For each positive integer ℓ , let $g(\ell) = \lfloor 2^\ell + 2^{(\ell+5)/2} + 10 \rfloor + 1$. Lemma 4.6 implies that $\mathfrak{K}(m) < g(\ell)$ for all odd positive integers $m < 2^\ell$. It follows from the definition of γ that $\gamma(g(\ell)) \geq 2^\ell + 1$. Therefore,

$$\limsup_{k \rightarrow \infty} \frac{\gamma(k)}{k} \geq \limsup_{\ell \rightarrow \infty} \frac{\gamma(g(\ell))}{g(\ell)} \geq \lim_{\ell \rightarrow \infty} \frac{2^\ell + 1}{2^\ell + 2^{(\ell+5)/2} + 11} = 1.$$

Now, choose an arbitrary positive integer k , and let $\ell = \lceil \log_2(\gamma(k)) \rceil$. By the definition of γ , $k < \mathfrak{K}(\gamma(k))$. We may use Lemma 4.6 to find that

$$\frac{\gamma(k)}{k} > \frac{\gamma(k)}{2^\ell + 2^{(\ell+5)/2} + 10} > \frac{2^{\ell-1}}{2^\ell + 2^{(\ell+5)/2} + 10}.$$

Note that this implies that $\gamma(k) \rightarrow \infty$ as $k \rightarrow \infty$. It follows that $\ell \rightarrow \infty$ as $k \rightarrow \infty$, so

$$\liminf_{k \rightarrow \infty} \frac{\gamma(k)}{k} \geq \lim_{\ell \rightarrow \infty} \frac{2^{\ell-1}}{2^\ell + 2^{(\ell+5)/2} + 10} = \frac{1}{2}.$$

□

In our final theorem, we provide upper bounds for $\liminf_{k \rightarrow \infty} (\gamma(k)/k)$ and $\limsup_{k \rightarrow \infty} (\gamma(k)/k)$. This will complete our proof of all the asymptotic results mentioned in the introduction. Before proving this theorem, we need one lemma. In what follows, recall that the Thue-Morse word \mathbf{t} is overlap-free. This means that if a, b, n are positive integers satisfying $a < b \leq a + n$, then $\langle a, a + n \rangle \neq \langle b, b + n \rangle$.

Lemma 4.7. *For each integer $\ell \geq 3$, we have*

$$\mathfrak{K}(3 \cdot 2^{\ell-2} + 1) > \frac{5 \cdot 2^{2\ell-3}}{3 \cdot 2^{\ell-2} + 1} \quad \text{and} \quad \mathfrak{K}(2^{\ell-1} + 3) > \frac{2^{2\ell-2}}{2^{\ell-1} + 3}.$$

Proof. Fix $\ell \geq 3$, and let $m = 3 \cdot 2^{\ell-2} + 1$ and $m' = 2^{\ell-1} + 3$. By the definitions of $\mathfrak{K}(m)$ and $\mathfrak{K}(m')$, there are nonnegative integers $r < \mathfrak{K}(m) - 1$ and $r' < \mathfrak{K}(m') - 1$ such that $\langle rm + 1, (r + 1)m \rangle = \langle (\mathfrak{K}(m) - 1)m + 1, \mathfrak{K}(m)m \rangle$ and $\langle r'm' + 1, (r' + 1)m' \rangle = \langle (\mathfrak{K}(m') - 1)m' + 1, \mathfrak{K}(m')m' \rangle$. According to Proposition 3.1, $2^{\ell-1}$ divides $(\mathfrak{K}(m) - 1)m - rm$ and $2^{\ell-2}$ divides $(\mathfrak{K}(m') - 1)m' - r'm'$. Since m and m' are odd, we know that $2^{\ell-1}$ divides $\mathfrak{K}(m) - r - 1$ and $2^{\ell-2}$ divides $\mathfrak{K}(m') - r' - 1$. If $\mathfrak{K}(m) - r - 1 \geq 2^\ell$, then $\mathfrak{K}(m) > \frac{5 \cdot 2^{2\ell-3}}{3 \cdot 2^{\ell-2} + 1}$ as desired. Therefore, we may assume $\mathfrak{K}(m) = r + 2^{\ell-1} + 1$. By the same token, we may assume that $\mathfrak{K}(m') = r' + 2^{\ell-2} + 1$.

With the aim of finding a contradiction, let us assume $\mathfrak{K}(m) \leq \frac{5 \cdot 2^{2\ell-3}}{m}$. Put

$$u = \langle rm + 1, (r + 1)m \rangle \quad \text{and} \quad v = \langle (\mathfrak{K}(m) - 1)m + 1, \mathfrak{K}(m)m \rangle.$$

We have

$$\mu^{2\ell-3}(01) = \mu^{2\ell-3}(\mathbf{t}_4\mathbf{t}_5) = \langle 3 \cdot 2^{2\ell-3} + 1, 5 \cdot 2^{2\ell-3} \rangle = wvz$$

for some words w and z . Observe that $|w| = (\mathfrak{K}(m) - 1)m - 3 \cdot 2^{2\ell-3} = rm + 2^{\ell-1}$. Since $\mu^{2\ell-3}(01) = \mu^{2\ell-3}(\mathbf{t}_1\mathbf{t}_2) = \langle 1, 2^{2\ell-3} \rangle$, we have $v = \langle rm + 2^{\ell-1} + 1, (r + 1)m + 2^{\ell-1} \rangle$. If we set $a = rm + 1$ and $b = rm + 2^{\ell-1} + 1$, then $a < b \leq a + m$. It follows from the fact that \mathbf{t} is overlap-free that $u \neq v$. This is a contradiction.

Assume now that $\mathfrak{K}(m') \leq \frac{2^{2\ell-2}}{m'}$. Let

$$u' = \langle r'm' + 1, (r' + 1)m' \rangle \quad \text{and} \quad v' = \langle (\mathfrak{K}(m') - 1)m' + 1, \mathfrak{K}(m')m' \rangle.$$

Let $q = \lceil (r'm' + 1)/2^{\ell-2} \rceil$ and $H = \min\{(r' + 1)m', (q + 2)2^{\ell-2}\}$. Finally, put $U = \langle r'm' + 1, H \rangle$ and $V = \langle (r' + 2^{\ell-2})m' + 1, H + 2^{\ell-2}m' \rangle$. The word U is the prefix of u' of length $H - r'm'$. Because $\mathfrak{K}(m') = r' + 2^{\ell-2} + 1$, V is the prefix of v' of length $H - r'm'$. Since $u' = v'$, we must have $U = V$.

There are words w' and z' such that

$$\mu^{\ell-2}(\mathbf{t}_q\mathbf{t}_{q+1}\mathbf{t}_{q+2}) = \langle (q - 1)2^{\ell-2} + 1, (q + 2)2^{\ell-2} \rangle = w'Uz'.$$

$\langle (q-1)2^{\ell-2} + 1, (q+2)2^{\ell-2} \rangle$				$\langle (q+m'-1)2^{\ell-2} + 1, (q+m'+2)2^{\ell-2} \rangle$		
$\mu^{\ell-2}(\mathbf{t}_q)$	$\mu^{\ell-2}(\mathbf{t}_{q+1})$	$\mu^{\ell-2}(\mathbf{t}_{q+2})$		$\mu^{\ell-2}(\mathbf{t}_{q+m'})$	$\mu^{\ell-2}(\mathbf{t}_{q+m'+1})$	$\mu^{\ell-2}(\mathbf{t}_{q+m'+2})$
w'	U		z'	w''	V	z''

FIGURE 2. An illustration of the proof of Lemma 4.7.

Furthermore,

$$\mu^{\ell-2}(\mathbf{t}_{q+m'}\mathbf{t}_{q+m'+1}\mathbf{t}_{q+m'+2}) = \langle (q+m'-1)2^{\ell-2} + 1, (q+m'+2)2^{\ell-2} \rangle = w''Vz''$$

for some words w'' and z'' . Note that $0 \leq |w'| = r'm' - (q-1)2^{\ell-2} = |w''| < 2^{\ell-2}$ (the inequalities follow from the definition of q). The suffix of $\mu^{\ell-2}(\mathbf{t}_q)$ of length $2^{\ell-2} - |w'|$ is a prefix of U . Similarly, the suffix of $\mu^{\ell-2}(\mathbf{t}_{q+m'})$ of length $2^{\ell-2} - |w''|$ is a prefix of V . Since $|w'| = |w''|$ and $U = V$, we must have $\mathbf{t}_q = \mathbf{t}_{q+m'}$. Similar arguments show that $\mathbf{t}_{q+1} = \mathbf{t}_{q+m'+1}$ and $\mathbf{t}_{q+2} = \mathbf{t}_{q+m'+2}$ (see Figure 2).

Now,

$$r' = \mathfrak{R}(m') - 2^{\ell-2} - 1 \leq \frac{2^{2\ell-2}}{m'} - 2^{\ell-2} - 1 = \frac{2^{2\ell-3} - 5 \cdot 2^{\ell-2} - 3}{m'},$$

so $\frac{r'm' + 1}{2^{\ell-2}} < 2^{\ell-1} - 5$. Therefore, $q + 4 < 2^{\ell-1}$. It follows that for each $j \in \{0, 1, 2\}$, the binary expansion of $q + m' + j - 1$ has exactly one more 1 than the binary expansion of $q + j + 2$. We find that $\mathbf{t}_{q+3}\mathbf{t}_{q+4}\mathbf{t}_{q+5} = \overline{\mathbf{t}_{q+m'}\mathbf{t}_{q+m'+1}\mathbf{t}_{q+m'+2}} = \overline{\mathbf{t}_q\mathbf{t}_{q+1}\mathbf{t}_{q+2}}$. However, utilizing the fact that \mathbf{t} is cube-free, it is easy to check that $X\overline{X}$ is not a factor of \mathbf{t} whenever X is a word of length 3. This yields a contradiction when we set $X = \mathbf{t}_q\mathbf{t}_{q+1}\mathbf{t}_{q+2}$. \square

Theorem 4.2. *Let $\gamma(k)$ be as in Definition 1.2. We have*

$$\liminf_{k \rightarrow \infty} \frac{\gamma(k)}{k} \leq \frac{9}{10} \quad \text{and} \quad \limsup_{k \rightarrow \infty} \frac{\gamma(k)}{k} \leq \frac{3}{2}.$$

Proof. For each positive integer ℓ , let $f(\ell) = \left\lfloor \frac{5 \cdot 2^{2\ell-3}}{3 \cdot 2^{\ell-2} + 1} \right\rfloor$ and $h(\ell) = \left\lfloor \frac{2^{2\ell-2}}{2^{\ell-1} + 3} \right\rfloor$. One may easily verify that $h(\ell) < f(\ell) \leq h(\ell+1)$ for all $\ell \geq 3$. Lemma 4.7 informs us that $\mathfrak{R}(3 \cdot 2^{\ell-2} + 1) > f(\ell)$. This means that the prefix of \mathbf{t} of length $(3 \cdot 2^{\ell-2} + 1)f(\ell)$ is an $f(\ell)$ -anti-power, so $\gamma(f(\ell)) \leq 3 \cdot 2^{\ell-2} + 1$. As a consequence,

$$\liminf_{k \rightarrow \infty} \frac{\gamma(k)}{k} \leq \liminf_{\ell \rightarrow \infty} \frac{\gamma(f(\ell))}{f(\ell)} \leq \lim_{\ell \rightarrow \infty} \frac{3 \cdot 2^{\ell-2} + 1}{f(\ell)} = \frac{9}{10}.$$

Now, choose an arbitrary integer $k \geq 3$. If $h(\ell) < k \leq f(\ell)$ for some integer $\ell \geq 3$, then the prefix of \mathbf{t} of length $(3 \cdot 2^{\ell-2} + 1)f(\ell)$ is an $f(\ell)$ -anti-power. This implies that $\gamma(k) \leq 3 \cdot 2^{\ell-2} + 1$, so

$$\frac{\gamma(k)}{k} < \frac{3 \cdot 2^{\ell-2} + 1}{h(\ell)}.$$

Alternatively, we could have $f(\ell) < k \leq h(\ell+1)$ for some $\ell \geq 3$. In this case, Lemma 4.7 tells us that the prefix of \mathbf{t} of length $(2^\ell + 3)h(\ell+1)$ is an $h(\ell+1)$ -anti-power. It follows that

$$\frac{\gamma(k)}{k} < \frac{2^\ell + 3}{f(\ell)}$$

in this case.

Combining the above cases, we deduce that

$$\limsup_{k \rightarrow \infty} \frac{\gamma(k)}{k} \leq \limsup_{\ell \rightarrow \infty} \left[\max \left\{ \frac{3 \cdot 2^{\ell-2} + 1}{h(\ell)}, \frac{2^{\ell+1} + 3}{f(\ell)} \right\} \right] = \max \left\{ \frac{3}{2}, \frac{6}{5} \right\} = \frac{3}{2}.$$

□

Remark 4.1. Preserve the notation from the proof of Theorem 4.2. We showed that

$$\frac{\gamma(k)}{k} < \frac{3 \cdot 2^{\ell-2} + 1}{h(\ell)} = \frac{3}{2} + o(1)$$

if $h(\ell) < k \leq f(\ell)$ and

$$\frac{\gamma(k)}{k} < \frac{2^{\ell} + 3}{f(\ell)} = \frac{6}{5} + o(1)$$

whenever $f(\ell) < k \leq h(\ell + 1)$ (the $o(1)$ terms refer to asymptotics as $k \rightarrow \infty$). This is indeed reflected in the top image of Figure 3, which portrays a plot of $\gamma(k)/k$ for $3 \leq k \leq 2100$.

5. CONCLUDING REMARKS

In Theorems 3.1 and 3.2, we obtained the exact values of $\liminf_{k \rightarrow \infty} (\Gamma(k)/k)$ and $\limsup_{k \rightarrow \infty} (\Gamma(k)/k)$. Unfortunately, we were not able to determine the exact values of $\liminf_{k \rightarrow \infty} (\gamma(k)/k)$ and $\limsup_{k \rightarrow \infty} (\gamma(k)/k)$. Figure 3 suggests that the upper bounds we obtained are the correct values.

Conjecture 5.1. *We have*

$$\liminf_{k \rightarrow \infty} \frac{\gamma(k)}{k} = \frac{9}{10} \quad \text{and} \quad \limsup_{k \rightarrow \infty} \frac{\gamma(k)}{k} = \frac{3}{2}.$$

Recall that we obtained lower bounds for $\liminf_{k \rightarrow \infty} (\gamma(k)/k)$ and $\limsup_{k \rightarrow \infty} (\gamma(k)/k)$ by first showing that $\mathfrak{R}(m) \leq 2^{\lceil \log_2 m \rceil} (1 + o(m))$. If Conjecture 5.1 is true, its proof will most likely require a stronger upper bound for $\mathfrak{R}(m)$.

We know from Theorem 3.1 that $(2\mathbb{Z}^+ - 1) \setminus \mathcal{F}(k)$ is finite whenever $k \geq 3$. A very natural problem that we have not attempted to investigate is that of determining the cardinality of this finite set. Similarly, one might wish to explore the sequence $(\Gamma(k) - \gamma(k))_{k \geq 3}$.

Recall that if w is an infinite word whose i^{th} letter is w_i , then $AP(w, k)$ is the set of all positive integers m such that $w_1 w_2 \cdots w_{km}$ is a k -anti-power. An obvious generalization would be to define $AP_j(w, k)$ to be the set of all positive integers m such that $w_{j+1} w_{j+2} \cdots w_{j+km}$ is a k -anti-power. Of course, we would be particularly interested in analyzing the sets $AP_j(\mathbf{t}, k)$.

Define a (k, λ) -anti-power to be a word of the form $w_1 w_2 \cdots w_k$, where w_1, w_2, \dots, w_k are words of the same length and $|\{i \in \{1, 2, \dots, k\} : w_i = w_j\}| \leq \lambda$ for each fixed $j \in \{1, 2, \dots, k\}$. With this definition, a $(k, 1)$ -anti-power is simply a k -anti-power. Let $\mathfrak{R}_\lambda(m)$ be the smallest positive integer k such that the prefix of \mathbf{t} of length km is not a (k, λ) -anti-power. What can we say about $\mathfrak{R}_\lambda(m)$ for various positive integers λ and m ?

Finally, note that we may ask questions similar to the ones asked here for other infinite words. In particular, it would be interesting to know other nontrivial examples of infinite words x such that $\min AP(x, k)$ grows linearly in k .

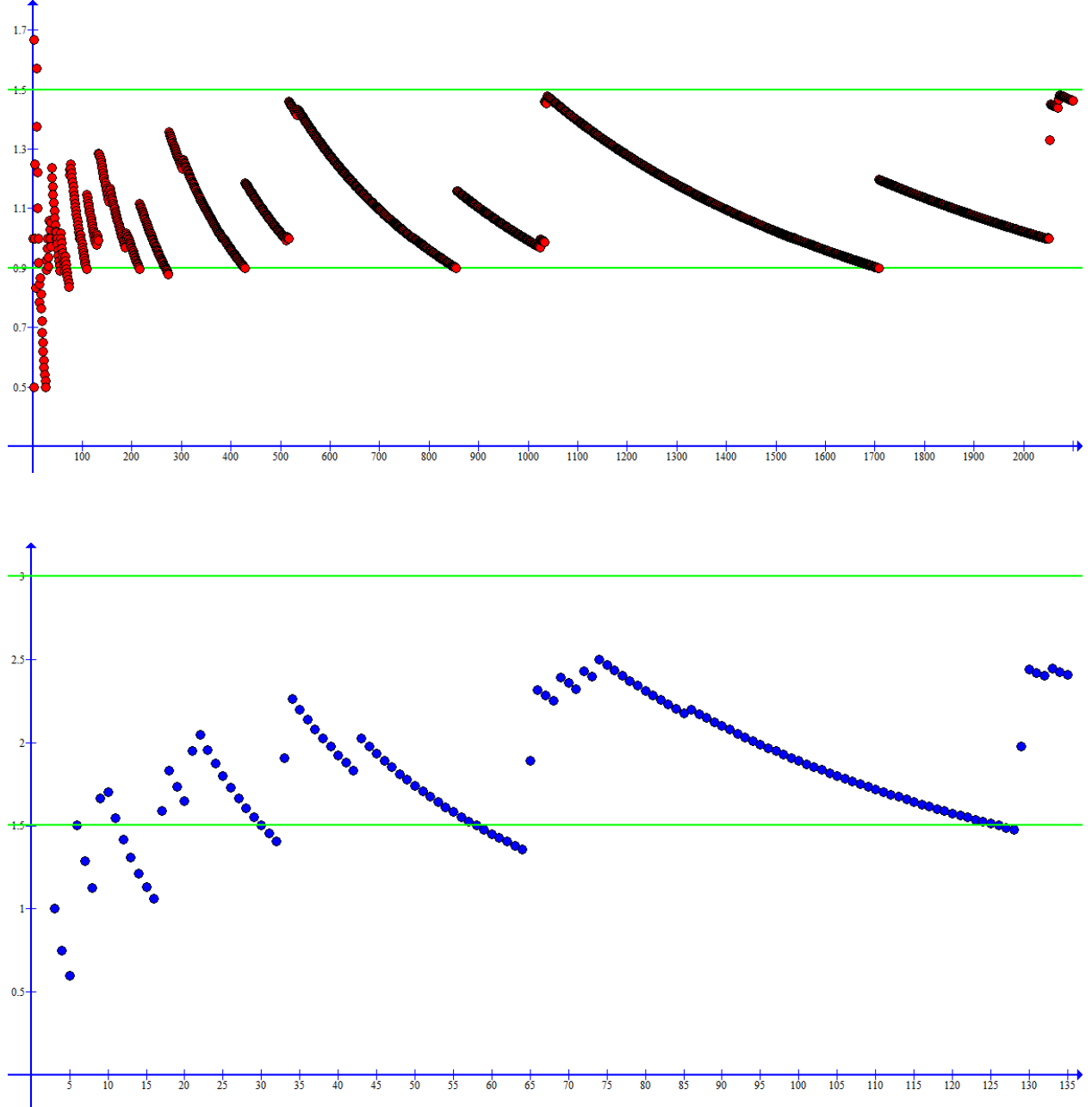


FIGURE 3. Plots of $\gamma(k)/k$ for $3 \leq k \leq 2100$ (top) and $\Gamma(k)/k$ for $3 \leq k \leq 135$ (bottom). In the top image, the green lines are at $y = 9/10$ and $y = 3/2$. In the bottom image, the green lines are at $y = 3/2$ and $y = 3$.

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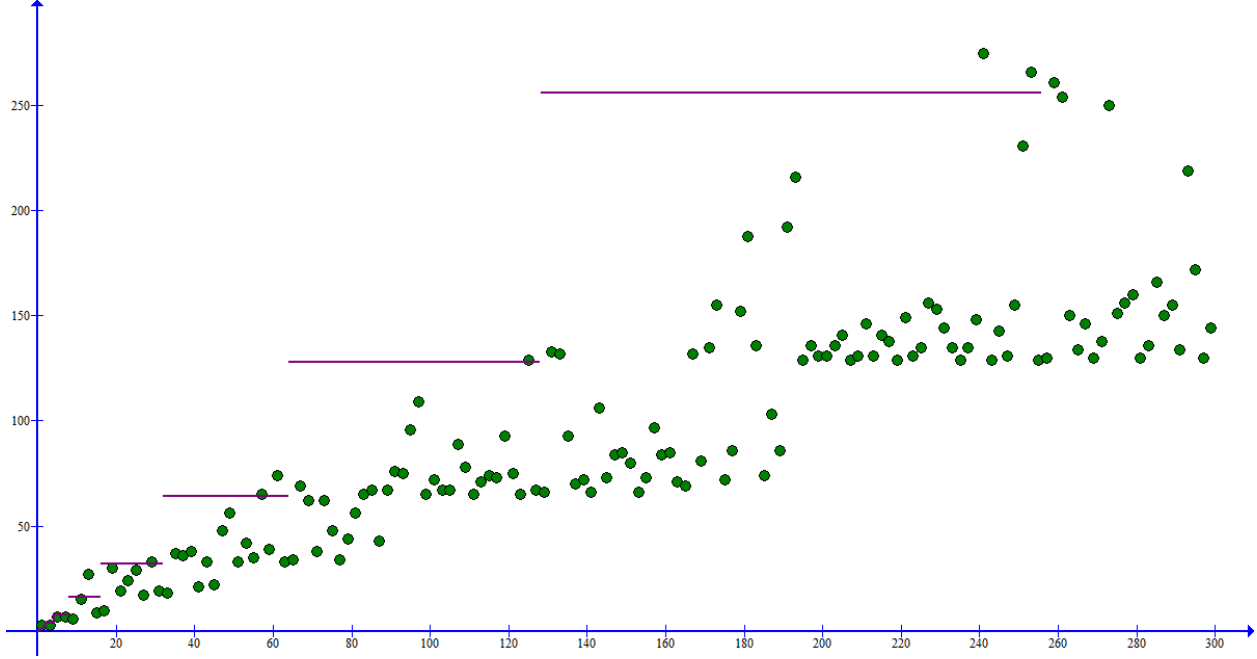


FIGURE 4. A plot of $\mathfrak{R}(m)$ for all odd positive integers $m \leq 299$. In purple is the graph of $y = 2^{\lceil \log_2 x \rceil}$.

REFERENCES

- [1] J.-P. Allouche, Thue, Combinatorics on words, and conjectures inspired by the Thue-Morse sequence, *J. de Théorie des Nombres de Bordeaux*, 27, no. 2 (2015), 375–388.
- [2] J.-P. Allouche and J. Shallit, The Ubiquitous Prouhet-Thue-Morse Sequence. *Sequences and their Applications: Proceedings of SETA '98*, (1999), 1–16.
- [3] Y. Bugeaud and G. Han, A combinatorial proof of the non-vanishing of Hankel determinants of the Thue-Morse sequence, *Electronic Journal of Combinatorics* 21(3) (2014).
- [4] S. Brlek, Enumeration of factors in the Thue-Morse word, *Discrete Applied Math.*, 24 (1989), 83–96.
- [5] F. Dejean, Sur un theoreme de Thue. *J. Combinatorial Theory Ser. A* 13 (1972), 90–99.
- [6] G. Fici, A. Restivo, M. Silva, and L. Zamboni, Anti-Powers in Infinite Words. *Proceedings of ICALP 2016*, (In Press).
- [7] A. Thue, Über die gegenseitige Lage gleich her Teile gewisser Zeichenreihen, *Kra. Vidensk. Selsk. Skifter. I. Mat.-Nat. Kl.*, Christiana 1912, Nr. 10.